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On Representation Spaces in Geometric Quantization[1]

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Abstract

The structure of the space of wave functions in the representation given by a complete strongly admissible polarization of the phase space is investigated. The conditions that the wave functions should be covariant constant along the real part of the polarization define the Bohr–Sommerfeld set of the representation containing the supports of all wave functions. There is a natural scalar product for the wave functions defined on the Bohr–Sommerfeld set. It is shown, for a real polarization, that the resulting Hilbert space of wave functions is not trivial if and only if the Bohr–Sommerfeld set is not empty.

1. *Introduction*

Let X denote a manifold representing the phase space of a classical system and ω a symplectic form on X defined by the Lagrange bracket. The canonical quantization of the system (X, ω) requires the notion of a classical counterpart of a complete set of independent commuting observables. As classical observables one could choose real-valued smooth functions f on X such that their Hamiltonian vector fields ξ_f, defined by

$$\xi_f \lrcorner\, \omega = -df \tag{1.1}$$

are complete. Therefore, a complete set of independent commuting observables would be a set of $n = \frac{1}{2} \dim X$ functions $f_1, \ldots, f_n$ on X, independent at all points of X, such that their Poisson brackets vanish, i.e.,

$$\omega(\xi_{f_i}, \xi_{f_j}) = 0, \qquad i, j = 1, \ldots, n \tag{1.2}$$

and the vector fields $\xi_{f_1}, \ldots, \xi_{f_n}$ are complete. However, for many phase spaces of interest, there does not exist such a set, and one has to relax the

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assumptions. If one drops the conditions that f_i's should be real and globally defined, one is led to notion of a polarization of (X, ω). Note that the vector fields $\xi_{f_1}, \cdots, \xi_{f_n}$ span over complex numbers an involutive complex distribution F on X such that $\dim_{\mathbb{C}} F = n$ and ω restricted to F vanishes identically.

The wave functions in the representation defined by a polarization F are "square integrable" sections of a certain complex line bundle over X, covariant constant along F. The condition that the wave functions should be covariant constant along the real part $F \cap \bar{F}$ of the polarization F defines a subset of the phase space, called the Bohr-Sommerfeld set of the representation, which contains the supports of all wave functions. The structure of the Bohr-Sommerfeld set for a complete strongly admissible polarization is studied here in detail. It is shown that there is an intrinsically defined scalar product in the space of sections of the complex line bundle defined over the Bohr-Sommerfeld set and covariant constant along the polarization. If the polarization is real, the resulting Hilbert space of wave functions is nontrivial if and only if the Bohr-Sommerfeld set is not empty. For partially complex polarizations, which correspond to a generalization of the representation in terms of analytic functions used by Bargmann, (1961), there could be additional conditions on the supports of the wave functions due to the requirement that they should be analytic in some variables. This problem requires further investigation.

The analysis of the structure of the space of wave functions is carried here within the framework of the theory of geometric quantization. This theory, introduced by Kostant (1970), provides a unified treatment of the construction of irreducible representations of connected Lie groups. On the other hand, it enables a theoretical physicist to carry the process of canonical quantization in such a way that all the structure used in the process is specified in geometric terms. This aspect of the theory extends the theory developed independently by Souriau (1970). The notion of a polarization of a symplectic manifold first appeared in the work of Auslander and Kostant (1971) on the representations of solvable Lie groups. The complex line bundle over X such that its sections covariant constant along the polarization are appropriate candidates for the wave functions is specified in the paper by Blattner (1973), summarizing the development of the theory of geometric quantization. The results obtained here are generalizations of the results of Śniatycki (1975), and Toporowski (1976).

The representation adopted in this paper is essentially self-contained as far as the theory of geometric quantization is concerned. However, we make use freely of the standard mathematical techniques of differentiable manifolds, topology, etc. The next section contains definitions and statement of the results. Proofs of the results obtained are contained in Section 3.

2. *Definitions and Results*

A *polarization* of a symplectic manifold (X, ω) is a complex involutive distribution F on X, i.e., an involutive subbundle of the complexified tangent bundle $\mathscr{T}^{\mathbb{C}}X$ of X such that ω restricted to F vanishes identically, $\omega \mid F \times F = 0$, and dimension of $F \cap \bar{F}$ is constant, where $\bar{F}$ denotes the complex conjugate of F. The complex distributions $F \cap \bar{F}$ and $F + \bar{F}$ defined by a polarization F are complexifications of some real distributions denoted by D and E, respectively,

$$F \cap \bar{F} = D^{\mathbb{C}}, \qquad F + \bar{F} = E^{\mathbb{C}} \tag{2.1}$$

Since D is the intersection of involutive distributions, $D = F \cap \bar{F} \cap \mathscr{T}X$, it is an involutive real distribution on X. Let X/D denote the space of integral manifolds of D and π_D: $X \to X/D$ the canonical projection. A polarization F is said to be *strongly admissible* if E is an involutive distribution and the spaces X/D and X/E of integral manifolds of D and E, respectively, are manifolds such that the canonical projections π_D: $X \to X/D$, π_E: $X \to X/E$, and π_{ED}: $X/D \to X/E$ are submersions.

Proposition 2.1. For a strongly admissible polarization F each integral manifold Λ of D has a canonically defined global parallelism such that the parallel vector fields commute. Parallel vector fields in Λ are the restrictions to Λ of the Hamiltonian vector fields in D.

A strongly admissible polarization F is said to be *complete* if the parallel vector fields on the integral manifolds of D are complete. For each positive integer k, we denote by $\mathbb{T}^k$ the k torus, i.e., the quotient of $\mathbb{R}^k$ by the discrete subgroup of $\mathbb{R}^k$ generated by the canonical basis in $\mathbb{R}^k$. The k torus has a canonically defined global parallelism in which parallel vector fields are the projections of constant vector fields in $\mathbb{R}^k$. Thus, the product $\mathbb{R}^l \times \mathbb{T}^k$ has canonically defined global parallelism and the parallel vector fields in $\mathbb{R}^l \times \mathbb{T}^k$ are complete.

Proposition 2.2. Let F be a complete strongly admissible polarization of (X, ω). For each $x \in X$ the integral manifold Λ of D through x is isomorphic, as a manifold with a global parallelism, to the product $\mathbb{R}^{d-k} \times \mathbb{T}^k$ for some integer k such that $0 \leqslant k \leqslant d = \dim D$.

Let, for each $k \in \{0, 1, \ldots, d\}$, X_k denote the subset of X consisting of all the points $x \in X$ such that the integral manifold of D through x is isomorphic to $\mathbb{R}^{d-k} \times \mathbb{T}^k$, and let $X^k = U_{l \geqslant k} X_l$. For each $x \in X$, we denote by K_x the subspace of D_x spanned by all $v \in D_x$ such that the parallel vector field, in the integral manifold of D through x, extending v is periodic. Clearly, for $x \in X_k$, $\dim K_x = k$. Let $K_x{}^{\perp}$ be the subspace of $\mathscr{T}_x X$ defined by

$$K_x{}^{\perp} = \{u \in \mathscr{T}_x X \mid \omega(u, v) = 0 \text{ for all } v \in K_x\} \tag{2.2}$$

Further, let $K = U_{x \in X} K_x$ and $K^{\perp} = U_{x \in X} K_x{}^{\perp}$. In general, neither K nor $K^{\perp}$ are distributions since their dimensions may vary from point to point.

Theorem 2.3. Let F be a complete strongly admissible polarization of (X, ω). For each $k \in \{0, 1, \ldots, d\}$ the following hold:

(A) X^k is open in X.

(B) If $x \in X_k$, then the integral manifold of E through x is contained in X_k.

(C) For each $y \in \pi_D(X_k)$ there exists a neighborhood V of y in X/D such that the following hold:

(i) For each $x \in \pi_D^{-1}(V) \cap X_k$ the integral manifold of E through x is contained in $\pi_D^{-1}(V)$.

(ii) There exists a canonical extension of $K \mid X_k \cap \pi_D^{-1}(V)$ to a k-dimensional involutive distribution ${}_VK$ on $\pi_D^{-1}(V)$ invariant under the action of the Hamiltonian vector fields in $E \mid \pi_D^{-1}(V)$, and contained in $K \mid \pi_D^{-1}(V)$. The integral manifolds of ${}_VK$ are diffeomorphic to $\mathbb{T}^k$. For $k \geqslant 1$, here exists a unique density ${}_V\kappa$ on ${}_VK$ invariant under the action of the Hamiltonian vector fields in $E \mid \pi_D^{-1}(V)$, and associating to each integral manifold of ${}_VK$ the total volume 1.

(iii) The distribution ${}_VK^{\perp}$ defined by

$$ {}_VK^{\perp} = \{u \in \mathscr{T}_xX \mid x \in \pi_D^{-1}(V),\ \omega(u, v) = 0 \text{ for all } v \in {}_VK_x\} \tag{2.3} $$

is a k-codimensional involutive distribution extending $K^{\perp} \mid X_k \cap \pi_D^{-1}(V)$ and it projects to an involutive k-codimensional distribution $\mathscr{T}\pi_D({}_VK^{\perp})$ on V.

Let L denote a complex line bundle over X with a connection ∇ such that the curvature form of ∇ is equal to the pull back of $-h^{-1}\omega$, where h denotes Planck's constant, and with a ∇ invariant Hermitian form $\langle , \rangle$. Such a line bundle exists if and only if $h^{-1}\omega$ defines an integral de Rham cohomology class and, if this condition is satisfied, the set of all equivalence classes of line bundles with connection having this curvature form can be parametrized by the group of all unitary characters of the fundamental group of X (Kostant, 1970).

Let $\mathscr{B}F$ denote the bundle of linear frames of F, where elements of $\mathscr{B}F$ are ordered n-tuples of linearly independent vectors in F_x, for some $x \in X$. The bundle $\mathscr{B}F$ is a right principal $Gl(n, \mathbb{C})$ fiber bundle over X, The nth exterior product of F is a complex line bundle $\wedge^n F$ over X. It is a fiber bundle associated to $\mathscr{B}F$ with the typical fibre $\mathbb{C}$ on which $Gl(n, \mathbb{C})$ acts by multiplication by the determinant of the corresponding matrix. We shall need a complex line bundle $\sqrt{\wedge^n F}$ defined as follows: Let $Ml(n, \mathbb{C})$ denote the double covering group of $Gl(n, \mathbb{C})$ and ρ: $Ml(n, \mathbb{C}) \to Gl(n, \mathbb{C})$ the covering homomorphism. A bundle of metalinear frames of F is a right principal $Ml(n, \mathbb{C})$ fiber bundle $\tilde{\mathscr{B}}F$ over X together with a map τ: $\tilde{\mathscr{B}}F \to \mathscr{B}F$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{\mathscr{B}}F \times Ml(n, \mathbb{C}) & \to & \tilde{\mathscr{B}}F \\ \Big\downarrow \tau \times \rho & & \Big\downarrow \tau \\ \mathscr{B}F \times Gl(n, \mathbb{C}) & \to & \mathscr{B}F \end{array}$$

where the horizontal arrows denote the group actions. The existence of a metalinear frame bundle of F is equivalent to the vanishing of a class in $H^2(X, \mathbb{Z}_2)$ characteristic of $\mathscr{B}F$. Let χ: $Ml(n, \mathbb{C}) \to \mathbb{C}$ be the unique holomorphic square root of the complex character $\det \circ \rho$ of $Ml(n, \mathbb{C})$ such that $\chi(\tilde{I}) = 1$, where $\tilde{I}$ denotes the identity of $Ml(n, \mathbb{C})$. The bundle $\sqrt{\Lambda^n F}$ is the fiber bundle over X associated to $Ml(n, \mathbb{C})$ with standard fibre $\mathbb{C}$ on which $Ml(n, \mathbb{C})$ acts by multiplication by $\chi(\tilde{A})$, $\tilde{A} \in Ml(n, \mathbb{C})$. In geometric quantization, one works traditionally with the bundle $(\Lambda^n)^{1/2}(TX/F)^*$ of half-forms normal to F, introduced by Blattner (1973), which is isomorphic to $\sqrt{\Lambda^n \bar{F}}$. In Śniatycki (1975) a bundle isomorphic to the dual of $\sqrt{\Lambda^n F}$ was used. We find it more convenient to work with $\sqrt{\Lambda^n F}$. The space of sections ν of $\sqrt{\Lambda^n F}$ is isomorphic to the space of functions $\nu^\# : \tilde{\mathscr{B}}F \to \mathbb{C}$ satisfying the condition $\nu^\#(\tilde{b} \cdot \tilde{A}) = \chi(\tilde{A}^{-1})\nu^\#(\tilde{b})$, for each $\tilde{b} \in \tilde{\mathscr{B}}F$ and each $\tilde{A} \in Ml(n, \mathbb{C})$. A strongly admissible polarization can be locally spanned by Hamiltonian vector fields (Nirenberg, 1957). A local section β of $\tilde{\mathscr{B}}F$ is called a *Hamiltonian metalinear frame field* of F if its projection to a section of $\mathscr{B}F$ consists of an ordered n-tuple of Hamiltonian vector fields spanning F. There is a canonically defined operator ∇ of partial covariant differentiation of sections of $\sqrt{\Lambda^n F}$ in direction F such that a section ν of $\sqrt{\Lambda^n F}$ is covariant constant along F if, for each Hamiltonian metalinear frame field β of F, the function $\nu^\# \circ \beta$ is constant along F (Simms, 1974).

The connection in L and the operator of partial covariant differentation of sections of $\sqrt{\Lambda^n F}$ in the direction F define an operator of partial covariant differentiation of sections of $L \otimes \sqrt{\Lambda^n F}$ in the direction F which will be denoted by ∇. For each integral manifold Λ of D the operator ∇ induces a flat connection in $(L \otimes \sqrt{\Lambda^n F}) \mid \Lambda$. The *Bohr-Sommerfeld set* S is defined as the union of all integral manifolds Λ of D such that the holonomy group of the flat connection in $(L \otimes \sqrt{\Lambda^n F}) \mid \Lambda$ vanishes. If $\lambda \otimes \nu$ is a global section of $L \otimes \sqrt{\Lambda^n F}$ covariant constant along F, then it is covariant constant along D so that $\lambda(x) \otimes \nu(x) = 0$ for $x \notin S$. For each $k \in \{0, 1, \ldots, d\}$, we denote by S_k the intersection of S with X_k, $S_k = S \cap X_k$.

Theorem 2.4. Let F be a complete strongly admissible polarization of (X, ω) and S the Bohr-Sommerfeld set defined by the operator ∇ of partial covariant differentiation of sections of $L \otimes \sqrt{\Lambda^n F}$ in the direction F. The following hold:

(A) S is closed.

(B) $S_0 = X_0$ and, for each $k \in \{1, \ldots, d\}$ and each $x \in S_k$, the integral manifold of E through x is contained in S_k.

(C) Given any point $x \in S_k$, let V be a neighborhood of $\pi_D(x)$ in X/D admitting a canonical extension of $K \mid X_k \cap \pi_D^{-1}(V)$ to a distribution ${}_VK$ on $\pi_D^{-1}(V)$ [cf. Theorem 2.3(C)], and let ${}_VK^\perp$ be the involutive distribution on $\pi_D^{-1}(V)$ defined as in Theorem 2.3. If Q denotes the integral manifold of $\mathscr{T}\pi_D({}_VK^\perp)$ passing through $\pi_D(x)$, then there exists an open set $\tilde{V}$ in X/D such that $Q \subseteq \tilde{V} \subseteq V$ and

(a) $S \cap \pi_D^{-1}(\tilde{V}) \subseteq \pi_D^{-1}(Q)$
(b) $X_k \cap \pi_D^{-1}(Q) \subseteq S_k$.

For each $k \in \{0, 1, \ldots, d\}$ consider the space of (discontinuous) sections $\lambda \otimes \nu$ of $L \otimes \sqrt{\Lambda^n F}$ satisfying the following conditions:

I. Support of $\lambda \otimes \nu$ is contained in S_k.

II. For each $x \in S_k$ the restriction of $\lambda \otimes \nu$ to $\pi_D^{-1}(Q)$, where Q is the integral manifold through $\pi_D(x)$ of the distribution $\mathscr{T}\pi_D({}_VK^\perp)$, for some neighborhood V of $\pi_D(x)$, is a smooth section of $(L \otimes \sqrt{\Lambda^n F}) \mid \pi_D^{-1}(Q)$ covariant constant along $F \mid \pi_D^{-1}(Q) \cap S_k$.

III. The projection to X/E of the support of $\lambda \otimes \nu$ has a finite number of connected components.

This space has a canonically defined pre-Hilbert subspace $\mathscr{H}_0{}^k$ with a scalar product $(\cdot|\cdot)_k$ described below. Let $\lambda_1 \otimes \nu_1$ and $\lambda_2 \otimes \nu_2$ be two sections satisfying the conditions I, II, and III. Then, there exists a finite number of open sets $\pi_D^{-1}(V_1), \ldots, \pi_D^{-1}(V_s)$ in X such that each $\pi_D^{-1}(V_i)$ admits a distribution ${}_{V_i}K$ and the supports of $\lambda_1 \otimes \nu_1$ and $\lambda_2 \otimes \nu_2$ are contained in the union of sets of the form $\nu_D^{-1}(Q_i)$, $i \in \{1, \ldots, s\}$, where $Q_1, \ldots, Q_s$, are integral manifolds of $\mathscr{T}\pi_D({}_{V_1}K), \ldots, \mathscr{T}\pi_D({}_{V_s}K^\perp)$, respectively. For each $i \in \{1, \ldots, s\}$ we associate to the pair of sections $\lambda_1 \otimes \nu_1$ and $\lambda_2 \otimes \nu_2$ a density $\langle \lambda_1 \otimes \nu_1, \lambda_2 \otimes \nu_2 \rangle_{Q_i}$ on Q_i in the way given in Śniatycki (1974). Let $x \in \pi_D^{-1}(Q_i)$ and consider a basis in $\mathscr{T}_x{}^{\mathbb{C}}X$ of the form

$$(v_1, \ldots, v_d, u_1, \ldots, u_{n-d}, \bar{u}_1, \ldots, \bar{u}_{n-d}, w_1, \ldots, w_d) \tag{2.4}$$

satisfying the following conditions:

(1) For each $j, l \in \{1, \ldots, d\}$ and $m \in \{1, \ldots, n-d\}$

$$\omega(v_j, w_l) = \delta_{jl}, \qquad \omega(u_m, w_l) = 0, \qquad \omega(w_j, w_l) = 0$$

(2) $(v_1, \ldots, v_k)$ is a basis in ${}_{V_i}K_x$ such that ${}_{V_i}\kappa(v_1, \ldots, v_k) = 1$.

(3) $(v_1, \ldots, v_d)$ is a basis in D_x.

(4) $b = (v_1, \ldots, v_d, u_1, \ldots, u_{n-d})$ is a basis in F_x and, for each $j, l \in \{1, \ldots, n-d\}$, $|\omega(u_j, \bar{u}_l)| = \delta_{jl}$, where $\bar{u}_l$ denotes the complex conjugate of u_l.

(5) $(v_1, \ldots, v_d, u_1, \ldots, u_{n-d}, \bar{u}_1, \ldots, \bar{u}_{n-d}, w_{k+1}, \ldots, w_d)$ is a basis in $\mathscr{T}_x{}^{\mathbb{C}}\pi_D^{-1}(Q_i)$.

This basis yields a basis

$$\begin{aligned}(\mathscr{T}\pi_D(u_1), \ldots, \mathscr{T}\pi_D(u_{n-d}), \mathscr{T}\pi_D\bar{u}_1), \ldots, \mathscr{T}\pi_D(\bar{u}_{n-d}),\\ \mathscr{T}\pi_D(w_{k+1}), \ldots, \mathscr{T}\pi_D(w_d))\end{aligned} \tag{2.5}$$

in $\mathscr{T}^{\mathbb{C}}Q_i$ at $\pi_D(x)$. Let $\tilde{b} \in \tilde{\mathscr{B}}F_x$ be a metalinear frame of F at x projecting to $b = (v_1, \ldots, v_d, u_1, \ldots, u_{n-d})$. The value of the density $\langle \lambda_1 \otimes \nu_1, \lambda_2 \otimes \nu_2 \rangle_{Q_i}$ on the basis (2.5) in $\mathscr{T}^{\mathbb{C}}Q_i$, at $\pi_D(x)$ is defined to be

$$\langle \lambda_1(x), \lambda_2(x) \rangle\, \nu_1{}^{\#}(\tilde{b})\, \overline{\nu_2{}^{\#}(\tilde{b})} \tag{2.6}$$

Following an argument similar to that in Śniatycki (1975) one shows that the expression (2.6) depends only on the basis (2.5) in $\mathscr{T}^{\mathbb{C}}Q_i$ at $\pi_D(x)$ in such a way that, under the change from the basis (2.4) to another basis in $\mathscr{T}^{\mathbb{C}}Q_i$ at $\pi_D(x)$ which comes from a basis in $\mathscr{T}_x{}^{\mathbb{C}}X$ satisfying conditions (1)–(5), it transforms as a density. Hence, $\langle\lambda_1\otimes\nu_1, \lambda_2\otimes\nu_2\rangle_{Q_i}$ defined by (2.6) is a density on Q_i. It is a smooth density since one can choose locally smooth linear frame fields for $\mathscr{T}^{\mathbb{C}}X$ satisfying conditions (1)–(5) and lift the resulting local linear frame field of F to a smooth local metalinear frame field of F.

Let $\mathscr{H}_0{}^k$ denote the subspace of the space of sections of $L\otimes\sqrt{\Lambda^n F}$ satisfying conditions I, II, and III and such that, for $\lambda\otimes\nu\in\mathscr{H}_0{}^k$

$$\|\lambda\otimes\nu\|^2 = \sum_i \int \langle\lambda\otimes\nu, \lambda\otimes\nu\rangle_{Q_i} < \infty \tag{2.7}$$

where, for each i, the integral is taken over the manifold Q_i and the summation is taken over all distinct manifolds Q_i such that $\pi_D^{-1}(Q_i)$ intersects the support of $\lambda\otimes\nu$. The scalar product in $\mathscr{H}_0{}^k$ is defined by

$$(\lambda_1\otimes\nu_1 \mid \lambda_2\otimes\nu_2)_k = \sum_i \int \langle\lambda_1\otimes\nu_1, \lambda_2\otimes\nu_2\rangle_{Q_i} \tag{2.8}$$

Clearly, this scalar product defines in $\mathscr{H}_0{}^k$ the pre-Hilbert space structure. Let $\mathscr{H}^k$ denote the Hilbert space obtained by the completion of $\mathscr{H}_0{}^k$ with respect to the norm (2.7). The space

$$\mathscr{H} = \otimes_{k=0}^{d}\mathscr{H}^k \tag{2.9}$$

is the Hilbert space of the wave functions in the representation given by the complete strongly admissible polarization F.

> *Theorem 2.5.* Let F be a complete strongly admissible polarization such that $F=\bar{F}$. Then, $\mathscr{H}\neq 0$ if and only if the Bohr–Sommerfeld set S is not empty.

3. *Proofs*

Let F be a strongly admissible polarization of (X, ω). Let $f_1, \ldots, f_d$ be coordinate functions defined in an open set U in X/E, where $d = \dim D = \dim X/E$. For each $i\in\{1, \ldots, d\}$ we denote by ξ^i the Hamiltonian vector field of the function $f_i\circ\pi_E$ on $\pi_E^{-1}(U)$. The vector fields $\xi^1, \ldots, \xi^d$ commute and span $D\,|\,\pi^{-1}(U)$. Hence, for each integral manifold Λ of D contained in $\pi_E^{-1}(U)$, the restrictions to Λ of $\xi^1, \ldots, \xi^d$ are commuting vector fields that span the tangent bundle space of Λ. This defines in Λ a global parallelism such that the parallel vector fields in Λ commute and they are the restrictions to Λ of the Hamiltonian vector fields in D. Clearly, the global parallelism in Λ defined here is independent of the choice of chart in X/E. Thus, Proposition 3.1 has been proved.

Let F be a complete strongly admissible polarization. Then, the Hamiltonian vector fields $\xi^1, \ldots, \xi^d$ in $D \mid \pi_E^{-1}(U)$ are complete. Let, for each $i \in \{1, \ldots, d\}$, $\varphi_t{}^i$ denote the one-parameter group of diffeomorphisms of $\pi_E^{-1}(U)$ generated by ξ^i. The commutativity of the vector fields $\xi^1, \ldots, \xi^d$ implies that the diffeomorphisms $\varphi_t{}^i$ and $\varphi_s{}^j$ commute, for each $i, j \in \{1, \ldots, d\}$ and $s, t \in \mathbb{R}$. Since ξ^i's are Hamiltonian vector fields, the one-parameter groups $\varphi_t{}^i$ preserve $\omega \mid \pi_E^{-1}(U)$. Therefore, we have an action Ψ: $\mathbb{R}^d \times \pi_E^{-1}(U) \to \pi_E^{-1}(U)$ of $\mathbb{R}^d$ on $\pi_E^{-1}(U)$ preserving the symplectic form $\omega \mid \pi_E^{-1}(U)$. This action is defined by $\Psi((t_1, \ldots, t_d), x) = \varphi^1_{t_1} \circ \cdots \circ \varphi^d_{t_d}(x)$, for each $(t_1, \ldots, t_d) \in \mathbb{R}^d$ and each $x \in \pi_E^{-1}(U)$. For each integral manifold Λ of D contained in $\pi_E^{-1}(U)$, the action of $\mathbb{R}^d$ restricted to Λ is transitive on Λ. This follows from the facts that, for each $x \in \Lambda$, the mapping Ψ_x: $\mathbb{R}^d \to \Lambda$ defined by $\Psi_x(t_1, \ldots, t_d) = \Psi((t_1, \ldots, t_d), x)$ is a local diffeomorphism and Λ is connected. Hence, the integral manifold Λ of D passing through x is diffeomorphic to the quotient of $\mathbb{R}^d$ by the isotropy group G_x of x defined by $G_x = \{(t_1, \ldots, t_d) \in \mathbb{R}^d \mid \Psi((t_1, \ldots, t_d), x) = x\}$. Since G_x is a discrete subgroup of $\mathbb{R}^d$ it is generated by $k = \operatorname{rank} G_x$ linearly independent (over $\mathbb{R}$) elements of $\mathbb{R}^d$, where $0 \leqslant k \leqslant d$. Thus, $\mathbb{R}^d/G_x \cong \mathbb{R}^{d-k} \times \mathbb{T}^k$ and, therefore, Λ is diffeomorphic to $\mathbb{R}^{d-k} \times \mathbb{T}^k$, where $\mathbb{T}^k$ denotes the k torus. Note that the diffeomorphism $\Lambda \cong \mathbb{R}^{d-k} \times \mathbb{T}^k$ maps the parallel vector fields on Λ to parallel vector fields on $\mathbb{R}^{d-k} \times \mathbb{T}^k$. This completes the proof of Proposition 2.2.

> *Lemma 3.1.* Let F be a complete strongly admissible polarization. For each $x \in \pi_E^{-1}(U)$, there exist a neighborhood V of $\pi_D(x) \in X/D$ and a 1-form θ on $\pi_D^{-1}(V)$ invariant under the action of $\mathbb{R}^d$ such that $\omega \mid \pi_D^{-1}(V) = d\theta$. Given a basis $(v_1, \ldots, v_d)$ in D_x, one can choose θ so that $\theta(v_1), \ldots, (v_d)$ are preassigned numbers.

Proof. Let V be a contractible neighborhood of $\pi_D(x) \in X/D$ admitting a section σ: $V \to X$ of π_D such that $\sigma(\pi_D(x)) = x$, and contained in $\pi_{ED}^{-1}(U)$. For each $t = (t_1, \ldots, t_d) \in \mathbb{R}^d$, we denote by Ψ_t the diffeomorphism of $\pi_E^{-1}(U)$ onto itself defined by $\Psi_t(x') = \Psi(t, x') \equiv \Psi((t_1, \ldots, t_d), x')$ for every $x' \in \pi_E^{-1}(U)$. Then, for each $t \in \mathbb{R}^d$, $\Psi_t \circ \sigma$ is a section of π_D defined over V, and the family $\{\Psi_t \circ \sigma\}$ of sections of π_D defines a foliation of $\pi_D^{-1}(V)$ transversal to the fibers of π_D. We shall refer to the distribution tangent to this foliation as the horizontal distribution, while the distribution tangent to the fibers of π_D will be called vertical. Since V is contractible, the tangent bundle space of V admits a global trivialization. Let $\tilde{\eta}_1, \ldots, \tilde{\eta}_{2n-d}$ be $2n-d = \dim V$ of linearly independent vector fields on V, and $\eta_1, \ldots, \eta_{2n-d}$ the horizontal lifts to $\pi_D^{-1}(V)$ of $\tilde{\eta}_1, \ldots, \tilde{\eta}_{2n-d}$, respectively. That is, $\eta_1, \ldots, \eta_{2n-d}$ are horizontal vector fields on $\pi_D^{-1}(V)$ invariant under the action of $\mathbb{R}^d$ and projecting to $\eta_1, \ldots, \eta_{2n-d}$, respectively. The collection $\xi^1, \ldots, \xi^d, \eta_1, \ldots, \eta_{2n-d}$ of vector fields on $\pi_D^{-1}(V)$ trivializes the tangent bundle space of $\pi_D^{-1}(V)$. Let $\tau_1, \ldots, \tau_d$ be 1-forms on $\pi_D^{-1}(V)$ defined by $\tau_i(\xi^j) = \delta_i{}^j$ and $\tau_i(\eta_l) = 0$, for all $i, j \in \{1, \ldots, d\}$, $l \in \{1, \ldots, 2n-d\}$. The forms $\tau_1, \ldots, \tau_d$ are invariant under the action of $\mathbb{R}^d$ and are closed, since the horizontal distribution spanned by the η_l's is involutive and $[\xi^i, \xi^j] = [\xi^i, \eta_l] = 0$, for all $i, j \in \{1, \ldots, d\}$, $l \in \{1, \ldots, 2n-d\}$. Consider the 2-form ω_1 in $\pi_D^{-1}(V)$ defined by $\omega_1 = \omega - d(\Sigma_{i=1}^d (f_i \circ \pi_E) \cdot \tau_i)$, where $f_1, \ldots, f_d$ are the coordinate functions on U defining $\xi^1, \ldots, \xi^d$ as the Hamiltonian vector fields of $f_1 \circ \pi_E, \ldots, f_d \circ \pi_E$, respectively. Clearly, ω_1 is a closed 2-form invariant under the action of $\mathbb{R}^d$ and such that, for each $i \in \{1, \ldots, d\}$, $\xi^i \rfloor \omega_1 = 0$. Hence, $\omega_1 = \pi_D{}^*\tilde{\omega}_1$ for a unique closed 2-form $\tilde{\omega}_1$ on V. Since V is contractible, each closed 2-form on V is exact and there exists a 1-form $\tilde{\theta}_1$ such that $\tilde{\omega}_1 = d\tilde{\theta}_1$. Then $\omega = d[\Sigma_{i=1}^d (f_i \circ \pi_E)\tau_i + \theta_1]$, where $\theta_1 = \pi_D{}^*\tilde{\theta}_1$. Let $(v_1, \ldots, v_d)$ be a basis in D_x and $a_1, \ldots, a_d$ arbitrary real numbers. We want to choose a 1-form θ on $\pi_D^{-1}(V)$ such that $\omega = d\theta$ and $\theta(v_i) = a_i$, for each $i \in \{1, \ldots, d\}$. Decomposing the vectors $\xi^1(x), \ldots, \xi^d(x)$ in terms of the basis $(v_1, \ldots, v_d)$ we get $\xi^i(x) = \Sigma_{j=1}^d a_{ij} v_j$, for each $i \in \{1, \ldots, d\}$. Since the vectors $\xi^1(x), \ldots, \xi^d(x)$ form a basis in D_x, the conditions $\theta(v_i) = a_i$, for each $i \in \{1, \ldots, d\}$ are equivalent to $\theta(\xi^i(x)) = a_i'$, where $a_i' = \Sigma_{j=1}^d a_{ij} a_j$, for each $i \in \{1, \ldots, d\}$. Let $\theta = \Sigma_{i=1}^d \{a_i' \tau_i + [f_i \circ \pi_E - f_i(\pi_E(x))] \tau_i\} + \theta_1$. Then, $d\theta = \omega \mid \pi_D^{-1}(V)$ and $\theta(\xi^i(x)) = a_i'$, for each $i \in \{1, \ldots, d\}$, as required. ∎

Lemma 3.2. Let F be a complete strongly admissible polarization and let $v \in D_x$ be such that the parallel vector field extending v in the integral manifold of D through x is periodic with period 1. There exist a neighborhood V of $\pi_D(x)$ in X/D and a vector field ξ in $D \mid \pi_D^{-1}(V)$ such that $\xi(x) = v$ and the orbits of ξ are periodic with period 1. If $\omega \mid \pi_D^{-1}(V) = d\theta$, then ξ is the Hamiltonian vector field of a function f on $\pi_D^{-1}(V)$ defined as follows: For each $x' \in \pi_D^{-1}(V)$, $f(x')$ is the integral of θ over the orbit of ξ through x'. If θ is invariant under the action of d Hamiltonian vector fields spanning $D \mid \pi_D^{-1}(V)$, then $f = \theta(\xi)$.

Any two Hamiltonian vector fields ξ^1 and ξ^2 in $\pi_D^{-1}(V)$ with periodic orbits with period 1 and such that $\xi^1(x) = \xi^2(x)$ are equal in $\pi_D^{-1}(V_1)$ for some open set V_1 such that $\pi_D(x) \in V_1 \subseteq V$.

Proof. Let U be a coordinate neighborhood of $\pi_E(x)$ in X/E with coordinate functions $f_1, \ldots, f_d$ and the corresponding Hamiltonian vector fields $\xi^1, \ldots, \xi^d$ spanning $D \mid \pi_E^{-1}(U)$, and V_0 a neighborhood of $\pi_D(x)$ in X/D admitting a section σ: $V_0 \to X$ of π_D such that $\sigma(\pi_D(x)) = x$. We use the notation introduced at the beginning of this section.

Let Φ: $\mathbb{R}^d \times V_0 \to \pi_D^{-1}(V_0)$ denote the mapping defined by $\Phi((t_1, \ldots, t_d), y) = \Psi((t_1, \ldots, t_d), \sigma(y))$, for each $(t_1, \ldots, t_d) \in \mathbb{R}^d$ and each $y \in V_0$. Clearly, Φ is a local diffeomorphism of $\mathbb{R}^d \times V_0$ onto $\pi_D^{-1}(V_0)$. By shrinking V_0, if necessary, we can find an open neighborhood W of $O \in \mathbb{R}^d$ such that the restriction of Φ to $W \times V_0$ is a diffeomorphism onto $\Phi(W \times V_0)$. This implies that $W \cap G_{\sigma(y)} = \{0\}$ for each $y \in V_0$, where $G_{\sigma(y)}$ denotes the isotropy group of $\sigma(y)$ with respect to the action Ψ of $\mathbb{R}^d$ on $\pi_E^{-1}(U)$. Since the isotropy groups of all points of an integral manifold of D are the same, it follows that $W \cap G_{x'} = 0$ for all $x' \in \pi_D^{-1}(V_0)$.

Let $s_1, \ldots, s_d$ be the components of v with respect to the basis

$(\xi^1(x), \ldots, \xi^d(x))$ in D_x, and Λ denote the integral manifold of D through x. The parallel vector field on Λ extending v coincides with the restriction to Λ of the Hamiltonian vector field $s_1\xi^1 + \cdots + s_d\xi^d$. Since the orbit of $s_1\xi^1 + \cdots + s_d\xi^d$ through x is periodic with period 1, it follows that Ψ_s is a diffeomorphism of $\pi_E^{-1}(U)$ onto itself that keeps x fixed, where $s = (s_1, \ldots, s_d)$. The set $\Psi_s(\Phi(W \times V_0)) \cap \Phi(W \times V_0)$ contains x and is open in $\pi_D^{-1}(V_0)$. Hence, $(W \times V_0) \cap \Phi^{-1}(\Psi_s \circ \Phi(W \times V_0))$ is an open set in $\mathbb{R}^d \times V_0$ containing $(0, \pi_D(x))$, and there exist a neighborhood V of $\pi_D(x)$ contained in V_0 and a smooth mapping $\beta\colon V \to W \subseteq \mathbb{R}^d$ such that, for each $y \in V$, $\Psi_{\beta(y)}(\sigma(y)) = \Psi_s(\sigma(y))$. Therefore, for each $y \in V$,

$$\begin{aligned}\Psi(s - \beta(y), \sigma(y)) &= \Psi_s(\Psi_{-\beta(y)}(\sigma(y)))\\ &= \Psi_s(\Psi_{-s}(\sigma(y))) = \Psi(s - s, \sigma(y))\\ &= \Psi(0, \sigma(y)) = \sigma(y)\end{aligned}$$

Moreover, the isotropy groups of the points on an integral manifold D coincide. Hence, for each $x' \in \pi_D^{-1}(V)$, $s - \beta(\pi_D(x')) \in G_{x'}$. Let ξ^0 be the vector field in $D \mid \pi_D^{-1}(V)$ such that, for each $x' \in \pi_D^{-1}(V)$, the components of $\xi^0(x')$ with respect to the basis $(\xi^1(x'), \ldots, \xi^d(x'))$ in $D_{x'}$ are given by the components of $s - \beta(\pi_D(x')) \in \mathbb{R}^d$. Since $\xi^1, \ldots, \xi^d$ are smooth vector fields and $\beta \circ \pi_D$ is a smooth mapping, ξ^0 is a smooth vector field. For each $x' \in \pi_D^{-1}(V)$, $s - \beta(\pi_D(x')) \in G_{x'}$ which implies that the orbit of ξ^0 through x' is periodic. Let f_0 be the function on $\pi_D^{-1}(V)$ that associates to each $x' \in \pi_D^{-1}(V)$ the period of the orbit of ξ^0 through x'. The function f_0 is a continuous smooth nonvanishing function on $\pi_D^{-1}(V)$ constant on the orbits of ξ^0 and $f_0(x) = 1$. The vector field $\xi = (1/f_0)\xi^0$ is a smooth vector field on $\pi_D^{-1}(V)$ with periodic orbits with period 1 and such that $\xi(x) = v$.

Let θ be a 1-form on $\pi_D^{-1}(V)$ such that $\omega \mid \pi_D^{-1}(V) = d\theta$. Let, for each $x' \in \pi_D^{-1}(V)$, $\gamma_{x'}\colon [0, 1] \to \pi_D^{-1}(V)$ denote the integral curve of ξ passing through x' and $f(x') = \int_0^1 \gamma_{x'}{}^*\theta = \oint\theta$, where the last integral is taken over the orbit of ξ through x'. The function f defined in this way is a smooth function on $\pi_D^{-1}(V)$ and, for each vector field ζ on $\pi_D^{-1}(V)$ we have

$$\zeta \cdot f = \oint \mathscr{L}_\xi\theta = \oint [\zeta \mathbin{\lrcorner} d\theta + d(\theta(\zeta))] = \oint \zeta \mathbin{\lrcorner} \omega = \oint_0^1 \omega(\zeta[\gamma_{x'}(t)], \xi[\gamma_{x'}(t)])dt.$$

For each $i \in \{1, \ldots, d\}$, we have $\xi^i \cdot f(x') = 0$ since $\omega(\xi^i, \xi) = 0$. Let $a_1, \ldots, a_d$ denote the components of ξ with respect to the basis $\xi^1, \ldots, \xi^d$, $\xi = \sum_{i=1}^d a_i\xi^i$. The functions a_i, $i \in \{1, \ldots, D\}$, are constant along the orbits of ξ and, for each $l \in \{1, \ldots, 2n\text{–}d\}$,

$$\begin{aligned}\eta_l \cdot f(x') &= \int_0^1 \omega(\eta_l[\gamma_{x'}(t)], \xi[\gamma_{x'}(t)])dt\\ &= \int_0^1 \sum_{i=1}^d a_i(\gamma_{x'}(t))\omega(\eta_l[\gamma_{x'}(t)], \xi^i[\gamma_{x'}(t)])dt\end{aligned}$$

$$= \sum_{i=1}^{d} a_i(x') \int_0^1 \omega(\eta_l[\gamma_{x'}(t)], \xi^i[\gamma_{x'}(t)])dt$$

$$= \sum_{i=1}^{d} a_i(x')\omega(\eta_l(x'), \xi^i(x'))$$

since, for each $l \in \{1, \ldots, 2n\text{-}d\}$ and each $i \in \{1, \ldots, d\}$, $\omega(\eta_l, \xi^i) = \eta_l \cdot (f_i \circ \pi_E)$ is constant along the orbit of ξ. Hence, $\eta_l \cdot f = \omega(\eta_l, \xi)$, for all $l \in \{1, \ldots, 2n\text{-}d\}$. This, and $\xi^i \cdot f = 0 = \omega(\xi^i, \xi)$, for all $i \in \{1, \ldots, d\}$, imply that $\xi \lrcorner \omega = -df$. Hence, ξ is the Hamiltonian vector field of f. Assume now that θ is invariant under the action of the vector fields $\xi^1, \ldots, \xi^d$. Given $x' \in \pi_D^{-1}(V)$, the Hamiltonian vector field $\xi' = a_1(x')\xi^1 + \cdots + a_d(x')\xi^d$ agrees with ξ on the orbit of ξ through x' and $\mathscr{L}_{\xi'}\theta = 0$. Therefore, on the orbit of ξ through x' we have $\xi(\theta(\xi)) = \xi'\theta(\xi) = \mathscr{L}_{\xi'}\theta(\xi) + \theta([\xi', \xi]) = 0$. Hence, $\theta(\xi)$ is constant along the orbit of ξ through x' and the integral of θ over the orbit is equal to $\theta(\xi(x'))$ since the orbit has period 1. Thus, $f(x') = \theta(\xi(x'))$, for each $x' \in \pi_D^{-1}(V)$.

Let ξ and $\tilde{\xi}$ be two Hamiltonian vector fields in $D \mid \pi_D^{-1}(V)$ such that $\xi(x) = \tilde{\xi}(x) = v$ and the orbits of ξ and $\tilde{\xi}$ are periodic with period 1. We denote by φ_t the one-parameter group of diffeomorphism of $\pi_D^{-1}(V)$ generated by $\xi - \tilde{\xi}$. Then, $\varphi_t(x) = x$ for all $t \in \mathbb{R}$ and $\varphi_1(x') = x'$ for all $x' \in \pi_D^{-1}(V)$. If $c_1, \ldots, c_d$ denote the components of $\xi - \tilde{\xi}$ with respect to the basis $\xi^1, \ldots, \xi^d$ and c: $\pi_D^{-1}(V) \to \mathbb{R}^d$ is defined by $c(x') = (c_1(x'), \ldots, c_d(x'))$, then $\Psi[(c_1(x'), \ldots, c_d(x')), x'] = \varphi_1(x') = x'$ for each $x' \in \pi_D^{-1}(V)$. Hence, $c(x') \in G_{x'}$, for each $x' \in \pi_D^{-1}(V)$ and $c(x) = 0$. Since c is continuous and W is an open neighborhood of $O \in \mathbb{R}^d$ such that $W \cap G_{x'} = \{0\}$, for each $x' \in \pi_D^{-1}(V)$, $c^{-1}(W)$ is an open neighborhood of $\pi_D(x)$ in V such that $\xi = \tilde{\xi}$ on $\pi_D^{-1}(c^{-1}(W))$. ∎

Proposition 3.3. If F is a complete strongly admissible polarization of (X, ω) then, for each $k \in \{0, 1, \ldots, d\}$, X^k is open in X.

Proof. Since $X^0 = X$, it is open. Assume $k \geqslant 1$. Given a point $x \in X^k$, let $v_1, \ldots, v_k$ be linearly independent vectors in D_x such that the parallel vector fields extending $v_1, \ldots, v_k$ in the integral manifold of D through x have periodic orbits with period 1. By Lemma 3.2, there exist Hamiltonian vector fields $\xi^1, \ldots, \xi^k$ in D, defined in a neighborhood $\pi_D^{-1}(V)$ of x such that, for each $i = 1, \ldots, k$, $\xi^i(x) = v_i$ and the orbits of ξ^i are periodic with period 1. Since $\xi^1, \ldots, \xi^k$ are linearly independent at x, they are linearly independent in some neighborhood W of x. Then, for each $x' \in W$, $\dim K_{x'} \geqslant k$, which implies that $W \subseteq X^k$. Hence, X^k is open in X. ∎

Proposition 3.4. If F is a complete strongly admissible polarization then, for each $k \in \{0, 1, \ldots, d\}$ and each $x \in X_k$, the integral manifold M of E through x is contained in X_k.

Proof. The complex version of the Frobenius theorem (Nirenberg, 1957) ensures that, for each $x' \in M$ there exist a neighborhood W of x' in X, d real

Hamiltonian vector fields $\xi^1, \ldots, \xi^d$ spanning $D \mid W$ and $(n\text{-}d)$ complex Hamiltonian vector fields $\zeta^1, \ldots, \zeta^{n-d}$ such that, $\xi^1, \ldots, \xi^d, \zeta^1, \ldots, \zeta^{n-d}$ span $F \mid W$. Since the Hamiltonian vector fields in F commute and the Hamiltonian vector fields spanning $D \mid W$ can be canonically extended to complete Hamiltonian vector fields spanning $D \mid \pi_D^{-1}(\pi_D(W))$, we may assume, without loss of generality, that $W = \pi_D^{-1}(V)$ for a neighborhood V of $\pi_D(x')$ in X/D. Let, for each $i \in \{1, \ldots, n\text{-}d\}$, Re ζ^i and Im ζ^i denote the real Hamiltonian vector fields on $\pi_D^{-1}(V)$ equal to the real and the imaginary part of ζ^i, respectively. The vector fields Re $\zeta^1, \ldots,$ Re ζ^{n-d}, Im $\zeta^1, \ldots,$ Im ζ^{n-d} induce local one-parameter groups of local diffeomorphism of $\pi_D^{-1}(V)$ preserving $\omega \mid \pi_D^{-1}(V)$ and commuting with the action of $\mathbb{R}^d$ on $\pi_D^{-1}(V)$ induced by $\xi^1, \ldots, \xi^d$. Moreover, $\xi^1, \ldots, \xi^d$, Re $\zeta^1, \ldots,$ Re ζ^{n-d}, Im $\zeta^1, \ldots,$ Im ζ^{n-d} span $E \mid \pi_D^{-1}(V)$. Hence, there exists a neighborhood $\tilde{V}$ of $\pi_D(x')$ in X/D such that $\tilde{V} \subseteq V$ and, for each $x'' \in \pi_D^{-1}(\tilde{V}) \cap M$ there exists a local diffeomorphism of M mapping diffeomorphically the integral manifold of D through x' onto the integral manifold of D through x''. Hence, the set of all points in M such that the integral manifolds of D through these points are diffeomorphic to the integral manifold of D through a chosen point $x' \in M$ is open in M. Since M is connected, it follows that all integral manifolds of D contained in M are diffeomorphic to each other. Hence, if $x \in M \cap X_k$, it follows that $M \subseteq X_k$. ∎

Proposition 3.5. Let F be a complete strongly admissible polarization of (X, ω). For each $y_0 \in \pi_D(X_k) \subseteq X/D$ there exists a neighborhood V of y_0 in X/D such that the following hold:

(i) For each $x \in \pi_D^{-1}(V) \cap X_k$ the integral manifold of E through x is contained in $\pi_D^{-1}(V)$.

(ii) There exists a canonical extension of $K \mid X_k \cap \pi_D^{-1}(V)$ to a k-dimensional involutive distribution ${}_VK$ on $\pi_D^{-1}(V)$ invariant under the action of the Hamiltonian vector fields in $E \mid \pi_D^{-1}(V)$, and contained in $K \mid \pi_D^{-1}(V)$. The integral manifolds of ${}_VK$ are diffeomorphic to $\mathbb{T}^k$. For $k > 0$, there exists a unique density ${}_V\kappa$ on ${}_VK$ invariant under the action of the Hamiltonian vector fields in $E \mid \pi_D^{-1}(V)$, and associating to each integral manifold of ${}_VK$ the total volume 1.

(iii) The distribution ${}_VK^\perp$ defined by ${}_VK^\perp = \{u \in \mathscr{T}_xX \mid x \in \pi_D^{-1}(V), \omega(u, v) = 0 \text{ for all } v \in {}_VK_x\}$ is a k-codimensional involutive distribution extending $K^\perp \mid X_k \cap \pi_D^{-1}(V)$ and it projects to an involutive k-codimensional distribution $\mathscr{T}\pi_D({}_VK^\perp)$ on V.

Proof. For $k = 0$, $K \mid X_0$ is a zero-dimensional vector bundle over X_0 and $K^\perp \mid X_0 = \mathscr{T}X \mid X_0$, so that the statement of Proposition 3.5 is obvious. Assume $k \geqslant 1$. Since X^k is open in X and, for each $x \in X_k$, the integral manifold of E through x is contained in X^k, given $y_0 \in \pi_D(X_k)$ we can choose a coordinate neighborhood U of $\pi_{ED}(y_0)$ in X/E such that $\pi_E^{-1}(U) \subseteq X^k$. For each $y \in \pi_D(X_k) \cap \pi_{ED}^{-1}(U)$, there exists a neighborhood V_y of y in $\pi_{ED}^{-1}(U)$ and k linearly independent Hamiltonian vector fields ${}_y\xi^1, \ldots, {}_y\xi^k$ in $K \mid \pi_D^{-1}(V_y)$ such that their orbits are periodic with period 1. For each

$x \in \pi_D^{-1}(V_y \cap V_{\tilde{y}} \cap \pi_D(X_k))$, the ordered k-tuples of vectors $({}_y\xi^1(x), \ldots, {}_y\xi^k(x))$ and $({}_{\tilde{y}}\xi^{-1}(x), \ldots, {}_{\tilde{y}}\xi^k(x))$ form bases for K_x, and they are related by a nonsingular linear transformation, ${}_y\xi^i(x) = \Sigma_{j=1}^k a_j{}^i(x){}_{\tilde{y}}\xi^j(x)$. But then, there exists a neighborhood $\tilde{V}$ of $\pi_D(x)$ contained in $V_y \cap \tilde{V}_y$ such that the matric coefficients $a_j{}^i$ are constant on $\pi_D^{-1}(\tilde{V})$; cf. Lemma 3.2. Hence, there exists an open set $V_{y\tilde{y}}$ containing $(V_y \cup V_{\tilde{y}}) \cap \pi_D(X_k)$ such that the vector fields ${}_y\xi^1, \ldots, {}_y\xi^k$ and ${}_y\xi^1, \ldots, {}_y\xi^k$ are related by a nonsingular linear transformation in $\pi_D^{-1}(V_{y\tilde{y}} \cap V_y \cap V_{\tilde{y}})$. Therefore, the vector fields ${}_y\xi^i$ and ${}_y\xi^j$ define a k-dimensional distribution on $\pi_D^{-1}(V_{y\tilde{y}})$. Continuing this process we obtain an open set V containing $(\cup_y V_y) \cap \pi_D(X_k)$ such that the vector fields ${}_y\xi^1, \ldots, {}_y\xi^k$, for $y \in \pi_{ED}^{-1}(U) \cap \pi_D(X_k)$ span a k-dimensional distribution ${}_VK$ in $\pi_D^{-1}(V)$. Clearly, ${}_VK$ agrees with K on $\pi_D^{-1}(V) \cap X_k$, and for each $x \in \pi_D^{-1}(V) \cap X_k$, the integral manifold of E through x is contained in $\pi_D^{-1}(V)$. Since the Hamiltonian vector fields in D commute with the Hamiltonian vector fields in E, the distribution ${}_VK$ is involutive and it is invariant under the action of the Hamiltonian vector fields in $E \mid \pi_D^{-1}(V)$. Since the orbits of the vector fields ${}_y\xi^1, \ldots, {}_y\xi^k$ spanning ${}_VK \mid \pi_D^{-1}(V_y)$ are periodic with period 1, it follows that all integral manifolds of ${}_VK$ are diffeomorphic to $\mathbb{T}^k$. If κ is any nonvanishing density on ${}_VK$ normalized so that the total volume of each integral manifold of ${}_VK$ is 1, by averaging κ over the integral manifolds of ${}_VK$ we obtain a density ${}_V\kappa$ invariant under the action of the Hamiltonian vector fields in ${}_VK$ such that the total volume of each integral manifold of ${}_VK$ is 1. This implies that, for each $y \in \pi_D(X_k) \cap \pi_{ED}^{-1}(U)$, ${}_V\kappa({}_y\xi^1, \ldots, {}_y\xi^k)$ has integral values so that it is constant in $\pi_D^{-1}(V \cap V_y)$. Hence ${}_V\kappa$ is invariant under the action of the Hamiltonian vector fields in $E \mid \pi_D^{-1}(V)$.

Let ${}_VK^\perp$ be defined by ${}_VK^\perp = \{u \in \mathcal{T}_xX \mid x \in \pi_D^{-1}(V), \omega(u, v) = 0$ for all $v \in {}_VK_x\}$. For each $y \in \pi_D(X_k) \cap \pi_{ED}^{-1}(U)$ we denote by ${}_yf^1, \ldots, {}_yf^k$ functions on $\pi_D^{-1}(V_y)$ such that each ${}_y\xi^i$ is the Hamiltonian vector field of ${}_yf^i, i \in \{1, \ldots, k\}$. Then, ${}_VK^\perp \mid \pi_D^{-1}(V_y \cap V)$ is uniquely characterized by the following condition: For each $x \in \pi_D^{-1}(V), u \in {}_VK_x{}^\perp$ if and only if $u \cdot {}_yf^1 = u \cdot {}_yf^2 = \cdots = u \cdot {}_yf^k = 0$. Hence, ${}_VK^\perp$ is smooth and integrable, i.e., involutive. Moreover, ${}_VK \subseteq D \mid \pi_D^{-1}(V)$ implies $E \mid \pi_D^{-1}(V) \subseteq {}_VK^\perp$. Hence, ${}_VK^\perp$ projects to a k-codimensional involutive distribution $\mathcal{T}\pi_D({}_VK^\perp)$ on V. ∎

Since Theorem 2.3 consists of the statements of Propositions 3.3, 3.4, and 3.5, the proof of Theorem 2.3 is completed. Similarly, we shall split Theorem 2.4 into a series of propositions. First we need the following lemma.

Lemma 3.6. Let F be a complete strongly admissible polarization. For each $y \in X/D$, there exists a neighborhood V of y in X/D such that, for each Hamiltonian vector field ξ in $K \mid \pi_D^{-1}(V)$ with periodic orbits with period 1, the function associating to each $x \in \pi_D^{-1}(V)$ the element of the holonomy group of $\sqrt{\wedge^n F}$ defined by the orbit of ξ through x is constant on $\pi_D^{-1}(V)$ and it takes on values ± 1.

Proof. Using the same argument as in the proof of Proposition 3.4, we can choose a connected neighborhood V of y such that $F \mid \pi_D^{-1}(V)$ is a trivial vector bundle globally spanned by Hamiltonian vector fields. Then $\Lambda^n F \mid \pi_D^{-1}(V)$ is trivial and, for each integral manifold Λ of D contained in $\pi_D^{-1}(V)$ the canonical flat connection in $\Lambda^n F \mid \Lambda$ has vanishing holonomy group. Since $\sqrt{\Lambda^n F} \mid \Lambda$ is a double covering of $\Lambda^n F \mid \Lambda$, for each $\Lambda \subseteq \pi_D^{-1}(V)$, the holonomy group of the canonical flat connection in $\sqrt{\Lambda^n F} \mid \Lambda$ is the multiplicative group consisting of 1 and -1. Further, the parallel transport along the orbits of ξ yields a continuous function on $\pi_D^{-1}(V)$. Since this function takes on values ± 1, it must be constant. ∎

Proposition 3.7. For a complete strongly admissible polarization F, $S_0 = X_0$ and, for each $k \in \{1, \ldots, d\}$ and each $x \in S_k$, the integral manifold of E through x is contained in S_k.

Proof. By Proposition 3.4, the integral manifold of E passing through $x \in X_k$ is contained in X_k. Since all integral manifolds of D contained in X_0 are contractible, it follows that the holonomy group of the flat connection in $(L \otimes \sqrt{\Lambda^n F}) \mid \Lambda$ vanishes, for each integral manifold Λ of D contained in X_0. Hence $X_0 = S_0$.

Consider the case $k \geqslant 1$. By Lemmas 3.1 and 3.2, there exist a contractible neighborhood V_y of $\pi_D(x)$ in X/D such that $\omega \mid \pi_D^{-1}(V_y)$ is exact, and k linearly independent Hamiltonian vector fields ${}_y\xi^1, \ldots, {}_y\xi^k$ in $K \mid \pi_D^{-1}(V_y)$ with periodic orbits with period 1. Since $\omega \mid \pi_D^{-1}(V_y)$ is exact, it follows that the curvature form of the connection in $L \mid \pi_D^{-1}(V_y)$ is exact, so that $L \mid \pi_D^{-1}(V_y)$ is a trivial line bundle. Let L^* denote the bundle over X obtained from L by the removal of the zero section and $\mathbb{C}^*$ denote the multiplicative group of nonzero complex numbers. Triviality of $L \mid \pi_D^{-1}(V_y)$ implies that there exists a mapping $p\colon L^* \mid \pi_D^{-1}(V_y) \to \mathbb{C}^*$ and a 1-form θ on $\pi_D^{-1}(V_y)$ such that the connection form in $L^* \mid \pi_D^{-1}(V_y)$ is given by $\alpha = (1/2\pi i)(dp/p) - h^{-1}\pi^*\theta$, where $\pi\colon L^* \mid \pi_D^{-1}(V_y) \to \pi_D^{-1}(V_y)$ denotes the fiber bundle projection. The relation between ω and the curvature form of the connection in L implies $\omega \mid \pi_D^{-1}(V_y) = d\theta$. Therefore, by Lemma 3.2, for each $i \in \{1, \ldots, k\}$, the vector field ${}_y\xi^i$ is the Hamiltonian vector field of the function ${}_yf^i$ such that, for each $x' \in \pi_D^{-1}(V)$, ${}_yf^i(x')$ is equal to the integral of θ over the orbit of ${}_y\xi^i$ through x'. On the other hand, the element of the holonomy group of the connection ∇ in L corresponding to the parallel transport along the orbit of ${}_y\xi^j$ through $x' \in \pi_D^{-1}(V_y)$ is given by $\exp[(2\pi i/h)\,{}_yf^j(x')]$. Lemma 3.6 implies that we can choose V_y small enough so that the elements of the holonomy group of $\sqrt{\Lambda^n F}$ defined by the orbits of ${}_y\xi^1, \ldots, {}_y\xi^k$ are constants ${}_yc^1, \ldots, {}_yc^k$ which are equal to 1 or -1. The following relations are the consequence of the definition of S:

$$\{x' \in S \cap \pi_D^{-1}(V_y)\} \Rightarrow \{{}_yc^j \exp[(2\pi i/h){}_yf^j(x')] = 1, \text{ for all } j = 1, \ldots, k\} \tag{3.1}$$

and

$$\{x' \in X_k \cap \pi_D^{-1}(V_y) \text{ and } {}_yc^j \exp[(2\pi i/h){}_yf^j(x')] = 1 \text{ for all } j = 1, \ldots, k\} \Rightarrow x' \in S_k \tag{3.2}$$

Moreover, the functions ${}_yf^1, \ldots, {}_yf^k$ are independent of $\pi_D^{-1}(V_y)$ because their Hamiltonian vector fields are linearly independent and they are constant along $E \mid \pi_D^{-1}(V_y)$.

Let M denote the integral manifold of E passing through x. For each $\tilde{x} \in M$ we have a neighborhood $V_{\tilde{y}}$ of $\tilde{y} = \pi_D(\tilde{x}) \in X/D$ and k functions ${}_{\tilde{y}}f^1, \ldots, {}_{\tilde{y}}f^k$ defined on $\pi_D^{-1}(V_{\tilde{y}})$ which are constant along $E \mid \pi_D^{-1}(V_{\tilde{y}})$, independent, and satisfy the relations

$$\{x' \in S \cap \pi_D^{-1}(V_{\tilde{y}})\} \Rightarrow \{{}_{\tilde{y}}c^j \exp[(2\pi i/h)\, {}_{\tilde{y}}f^j(x')] = 1 \text{ for all } j = 1, \ldots, k\}$$

and

$$\{x' \in X_k \cap \pi_D^{-1}(V_{\tilde{y}}) \text{ and } {}_{\tilde{y}}c^j \exp[(2\pi i/h)\, {}_{\tilde{y}}f^j(x')] = 1 \quad \text{for all } j = 1, \ldots, k\} \Rightarrow x' \in S_k$$

Hence, $\tilde{x} \in S_k \cap M$ if and only if $M \cap \pi_D^{-1}(V_{\tilde{y}}) \subseteq S_k \cap M$ which implies that $M \cap S_k$ is open and closed in M in the induced topology. Since the manifold topology on M is finer than the induced topology and M is connected, it follows that $M \cap S_k$ is either empty or equal to M. By hypothesis $x \in M \cap S_k$ so that $M \cap S_k = M$. Hence, $M \subseteq S_k$. ∎

Proposition 3.8. If F is a complete strongly admissible polarization then the Bohr–Sommerfeld set S is closed in X.

Proof. If $S = X$, it is closed. If $S \neq X$, we show that $X - S$ is open. Let $x \in X - S$ and $k = \dim K_x$. Then $k > 0$ since $X_0 - S_0 = \emptyset$. Let ${}_yf^1, \ldots, {}_yf^k$ be functions defined in a neighborhood $\pi_D^{-1}(V_y)$ of x, where $y = \pi_D(x)$, satisfying (3.1) and (3.2). Since $x \notin S_k$, there exists $j \in \{1, \ldots, k\}$ such that ${}_yc^j \exp\{(2\pi i/h)\, {}_yf^j(x)\} \neq 1$. Then there exists a neighborhood W of x contained in $\pi_D^{-1}(V_y)$ such that ${}_yc^j \exp[(2\pi i/h)\, {}_yf^j(x')] \neq 1$ for all $x' \in W$. This and the relation (3.1) imply that $W \subseteq X - S$. Hence $X - S$ is open and S is closed. ∎

Proposition 3.9. Let F be a complete strongly admissible polarization, $x \in S_k$ and V a neighborhood of $\pi_D(x) \in X/D$ such that $\pi_D^{-1}(V)$ admits a canonical extension of $K \mid X_k \cap \pi_D^{-1}(V)$ to a distribution ${}_VK$ on $\pi_D^{-1}(V)$. Let Q be the integral manifold of $\mathscr{T}\pi_D({}_VK^\perp)$ passing through $\pi_D(x)$. Then, $X_k \cap \pi_D^{-1}(Q) \subseteq S_k$ and there exists an open set $\tilde{V}$ in X/D such that $Q \subseteq \tilde{V} \subseteq V$ and $S \cap \pi_D^{-1}(\tilde{V}) \subseteq \pi_D^{-1}(Q)$.

Proof. If $k = 0$, Q is a zero codimensional submanifold of $V \subseteq X/D$ so that Q is an open set in V containing y. Then we can take $\tilde{V} = Q$ and the conditions $Q \subseteq \tilde{V} \subseteq V$ and $S \cap \pi_D^{-1}(\tilde{V}) \subseteq \pi_D^{-1}(Q)$ are satisfied, and $X_0 \cap \pi_D^{-1}(Q) \subseteq S_0$ because $S_0 = X_0$. Let us assume now that $k \geqslant 1$. For each $y \in V$, there exists a neighborhood V_y of y contained in V and k functions ${}_yf^1, \ldots, {}_yf^k$ on $\pi_D^{-1}(V)$ satisfying the relations (3.1) and (3.2) in the proof of Proposition 3.7 and such that their Hamiltonian vector fields span ${}_VK \mid \pi_D^{-1}(V_y)$ and have periodic orbits with period 1. Further, ${}_VK^\perp \mid \pi_D^{-1}(V_y)$ is characterized by the condition: $u \in {}_VK^\perp \mid \pi_D^{-1}(V_y)$ if and only if $u \cdot {}_yf^1 = u \cdot {}_yf^2 = \cdots = u \cdot {}_yf^k$.

Therefore, the functions ${}_yf^1, \ldots, {}_yf^k$ are constant on integral manifolds of ${}_VK^\perp \mid \pi_D^{-1}(V_y)$. Since ${}_yc^1, \ldots, {}_yc^k$ are constant on $\pi_D^{-1}(V_y)$, it follows that ${}_yc^j \exp[(2\pi i/h)\, {}_yf^j]$ is constant on integral manifolds of ${}_VK^\perp \mid \pi_D^{-1}(V_y)$, for each $j \in \{1, \ldots, k\}$. In particular, $\pi_D^{-1}(Q)$ is the integral manifold of ${}_VK^\perp$ passing through x and it satisfies

$$ {}_yc^j \exp[(2\pi i/h)\, {}_yf^j] \mid \pi_D^{-1}(V_y \cap Q) = 1 \tag{3.3}$$

for each $j \in \{1, \ldots, k\}$ and each $y \in V$. This and (3.2) imply that $X_k \cap \pi_D^{-1}(Q) \subseteq S_k$. For each $y \in Q$, let W_y denote the open subset of $\pi_D^{-1}(V_y)$ defined by $x' \in W_y$ if and only if $|\,{}_yf^j(x') - {}_yf^j(\tilde{x})\,| < h/4\pi$ for each $j \in \{1, \ldots, k\}$, where $\tilde{x}$ is an arbitrary point of $\pi_D^{-1}(y)$. Since the functions ${}_yf^1, \ldots, {}_yf^k$ are constant along integral manifolds of $D \mid \pi_D^{-1}(V_y)$, $W_y = \pi_D^{-1}(\pi_D(W_y))$. Put $\tilde{V} = \cup_y \pi_D(W_y)$. Then $\cup_y W_y = \pi_D^{-1}(\tilde{V})$. For each $x' \in \pi_D^{-1}(\tilde{V}) - Q$, the definition of W_y's and the fact that the constants ${}_yc^j$ are equal to ± 1 imply that ${}_yc^j \exp[(2\pi i/h)\, {}_yf^j(x')] \neq 1$ for all $j \in \{1, \ldots, k\}$ and all y such that $x' \in W_y$. Clearly, $Q \subseteq \tilde{V} \subseteq V$, and (3.1) implies that $S \cap \pi_D^{-1}(\tilde{V}) \subseteq \pi_D^{-1}(Q)$. ∎

Propositions 3.7, 3.8, and 3.9 imply Theorem 2.4. Hence it remains to prove Theorem 2.5. First, we need the following lemma.

Lemma 3.10. Let F be a complete strongly admissible polarization of (X, ω). For each $x \in S_k - \operatorname{Int} X_k$, where $\operatorname{Int} X_k$ denotes the interior of X_k, and each neighborhood W of x, $W \cap X^{k+1} \cap S \neq \emptyset$.

Proof. Let $x \in S_k - \operatorname{Int} X_k \subseteq X_k - \operatorname{Int} X_k$. Since $X^k = X_k \cup X^{k+1}$ is open, each neighborhood of x has a nonempty intersection with X^{k+1}. Hence x belongs to the closure of X^{k+1}, $x \in \operatorname{Cl} X^{k+1}$. Further, $X^{k+1} = X_{k+1} \cup X^{k+2}$, so that either $x \in X^{k+2}$ or $x \in \operatorname{Cl} X_{k+1}$ and there exists a neighborhood W of x such that $W \cap X^{k+2} = \emptyset$. If $W \cap X^{k+2} = \emptyset$ then, since X^{k+1} is open, $W \cap X^{k+1} = W \cap (X_{k+1} \cup X^{k+2}) = W \cap X_{k+1}$ is an open set contained in X_{k+1} and containing x in its closure. Therefore, $x \in \operatorname{Cl}(W \cap X_{k+1}) \subseteq \operatorname{Cl}(\operatorname{Int} X_{k+1})$. Hence, $x \in \operatorname{Cl} X^{k+1}$ implies either $x \in \operatorname{Cl}(\operatorname{Int} X_{k+1})$ or $x \in X^{k+2}$. If $x \in \operatorname{Cl} X^{k+2}$, we can repeat this process to get $x \in \operatorname{Cl}(\operatorname{Int} X_{k+2})$ or $x \in \operatorname{Cl} X^{k+3}$, and so on. Since $X_d = X^d$ is open, it follows that there exists an integer $l > k$ such that $x \in \operatorname{Cl}(\operatorname{Int} X_l)$.

Let V_y be a neighborhood of $y = \pi_D(x)$ in X/D such that $\pi_D^{-1}(V_y)$ admits k functions ${}_yf^1, \ldots, {}_yf^k$ satisfying (3.1) and (3.2) in the proof of Proposition 3.7. The Hamiltonian vector fields ${}_y\xi^1, \ldots, {}_y\xi^k$ of ${}_yf^1, \ldots, {}_yf^k$, respectively, are in K and have periodic orbits with period 1. Since, for some $l > k$, $x \in \operatorname{Cl}(\operatorname{Int} X_l)$ there exists a contractible set $V \subseteq V_y \cap \pi_D(\operatorname{Int} X_l)$ containing y in its closure and l independent functions $g^1, \ldots, g^l$ on $\pi_D^{-1}(V)$ such that their Hamiltonian vector fields $\xi^1, \ldots, \xi^l$ are in K and have periodic orbits with period 1, for each $j \in \{1, \ldots, k\}$, $f^j \mid \pi_D^{-1}(V) = g^j$, and

$$\{x' \in S \cap \pi_D^{-1}(V)\} \Leftrightarrow c^j \exp[(2\pi i/h) g^j(x')] = 1 \qquad \text{for all } j = 1, \ldots, l \tag{3.4}$$

where $c^1, \ldots, c^l$ are the elements of the holonomy group of $\sqrt{\wedge^n F}$ defined by the orbits of $\xi^1, \ldots, \xi^l$, respectively, and $c^j = {}_yc^j$, for each $j \in \{1, \ldots, k\}$.

Suppose that, for each $j \in \{k+1, \ldots, l\}$, $g^j(x')$ is not bounded as $x' \to x$. Let Q be the integral manifold of $\mathscr{T}\pi_D({}_{V_y}K^\perp)$ passing through $\pi_D(x)$. For each neighborhood W of x, $\pi_D^{-1}(Q) \cap W \cap \pi_D^{-1}(V)$ is a k-codimensional submanifold of $W \cap \pi_D^{-1}(V)$. Since, by hypothesis, there exists a sequence $x_m \to x$ such that $|g^j(x_m)| \to \infty$, for $j \in \{k+1, \ldots, l\}$, and the restrictions to $\pi_D^{-1}(Q) \cap W \cap \pi_D^{-1}(V)$ of the functions $g^{k+1}, \ldots, g^l$ are real and continuous, the set

$$\{x' \in \pi_D^{-1}(Q) \cap W \cap \pi_D^{-1}(V) \mid c^j \exp[(2\pi i/h)g^j(x')] = 1 \text{ for all } j = k+1, \ldots, l\} \tag{3.5}$$

is not empty. However, the relations (3.3) and (3.4) imply that the set (3.5) is contained in S. Hence, for each neighborhood W of x, $S \cap W \cap \pi_D^{-1}(V) \neq \emptyset$, which implies that $S \cap W \cap X^{k+1} \neq \emptyset$.

It remains to prove that, for each $j \in \{k+1, \ldots, l\}$, $|g^j(x')|$ is unbounded as $x' \to x$. Without loss of generality, we may assume that $D \mid \pi_D^{-1}(V_y)$ is globally spanned by Hamiltonian vector fields so that we have an action of $\mathbb{R}^d$ on $\pi_D^{-1}(V_y)$ denoted by Ψ: $\mathbb{R}^d \times \pi_D^{-1}(V_y) \to \pi_D^{-1}(V_y)$ which preserves the restriction of ω to $\pi_D^{-1}(V_y)$. For each $x' \in \pi_D^{-1}(V_y)$ the integral manifold $\Lambda_{x'}$ of D through x' is diffeomorphic to the quotient of $\mathbb{R}^d$ by the isotropy group $G_{x'}$, and we denote by $\alpha_{x'}$: $\mathbb{R}^d/G_{x'} \to \Lambda_{x'}$ the canonical isomorphism. For each $j \in \{k+1, \ldots, l\}$ we define a function φ^j: $\pi_D^{-1}(V) \to \mathbb{R}^d$ by $\varphi^j(x') = \mathscr{T}\alpha_{x'}^{-1}(\xi^j(x'))$, for each $x' \in \pi_D^{-1}(V)$. Since the vector fields $\xi^{k+1}, \ldots, \xi^l$ have periodic orbits with period 1, $\varphi^j(x') \in G_{x'}$ for each $x' \in \pi_D^{-1}(V)$ and each $j \in \{k+1, \ldots, l\}$. We shall show first that $|\varphi^j(x')| \to \infty$ as $x' \to x$, for all $j \in \{k+1, \ldots, l\}$, where $|\varphi^j(x')|$ denotes the Euclidean norm of the vector $\varphi^j(x') \in \mathbb{R}^d$. Suppose that, for some $j \in \{k+1, \ldots, l\}$, $|\varphi^j(x')|$ does not tend to infinity as $x' \to x$. That is, there exists a constant $c > 0$ such that, for each neighborhood W of x, there exists $x' \in W$ satisfying $|\varphi^j(x')| < c$. Hence there exists a sequence $\{x_m\}$ in $\pi_D^{-1}(V)$ convergent to x such that $\varphi^j(x_m)$ converges to a vector $\lim \varphi^j(x) \in \mathbb{R}^d$. The continuity of the action Ψ of $\mathbb{R}^d$ on $\pi_D^{-1}(V_y)$ and the relation $\varphi^j(x_m) \in G_{x_m}$ for each $m \in \mathbb{Z}^+$ imply that $\lim \varphi^j(x) \in G_x$. Since G_x has rank k and the vectors ${}_y\xi^1(x), \ldots, {}_y\xi^k(x)$ span K_x, each element of G_x is a linear combination of $\mathscr{T}\alpha_x^{-1}(\xi^1(x)), \ldots, \mathscr{T}\alpha_x^{-1}(\xi^k(x))$ with rational coefficients so that $\lim \varphi^j(x) = \Sigma_{i=1}^k a_i \, \mathscr{T}\alpha_x^{-1}(\xi^i(x))$ for some rational numbers $a_1, \ldots, a_k$. On the other hand, for each $m \in \mathbb{Z}^+$, $\varphi^j(x_m)$ is linearly independent of $\mathscr{T}\alpha_{x_m}^{-1}(\xi^1(x_m)), \ldots, \mathscr{T}\alpha_{x_m}^{-1}(\xi^k(x_m))$. The action Ψ of $\mathbb{R}^d$ in $\pi_D^{-1}(V_y)$ includes a local diffeomorphism Φ: $\mathbb{R}^d \times V_1 \to \pi_D^{-1}(V_1)$, where V_1 is a neighborhood of y contained in V_y admitting a section σ: $V_1 \to \pi_D^{-1}(V_1)$ such that $\sigma(y) = x$. The mapping Φ is defined by $\Phi((t_1, \ldots, t_d), y') = \Psi((t_1, \ldots, t_d), \sigma(y'))$ for each $y' \in V_1$. Let $B \times V_2 \subseteq \mathbb{R}^d \times V_1$ be a neighborhood of $(0, y)$ such that $\Phi \mid B \times V_2$ is a diffeomorphism of $B \times V_2$ onto its image $\Phi(B \times V_2)$. Then, for each $x' \in \pi_D^{-1}(V_2)$, $G_{x'} \cap B = \{0\}$. Since $a_1, \ldots, a_k$ are rational numbers, there exists a positive integer N such that $Na_1, \ldots, Na_k$ are integers. Then, for each $m \in \mathbb{Z}^+$, the vector $s_m = N[\varphi^j(x_m) - \Sigma_{i=1}^k a_i \, \mathscr{T}\alpha_{x_m}^{-1}(\xi^i(x_m))]$ is different from zero, is contained in G_{x_m} and

$s_m \to 0$ as $m \to \infty$. Hence, for m large enough, $x_m \in \pi_D^{-1}(V_2)$ and $s_m \in B$, which leads to contradiction with the assumption that $G_{x'} \cap B = \{0\}$ for all $x' \in \pi_D^{-1}(V_2)$. Therefore, $|\varphi^j(x')| \to \infty$ as $x' \to x$, for each $j = k+1, \ldots, l$. The set $\{(\varphi^{k+1}(x')/|\varphi^{k+1}(x')|, \ldots, \varphi^l(x')/|\varphi^l(x')|) \mid x' \in \pi_D^{-1}(V)\}$ is bounded in $\mathbb{R}^{d(l-k)}$. Hence, there exists a sequence $x_m \to x$ and $l-k$ vectors $v^{k+1}, \ldots, v^l$ in D_x such that, for each $j \in \{k+1, \ldots, l\}$, $\xi^j(x_m)/|\varphi^j(x_m)| \to v^j$ as $m \to \infty$.

It follows from Lemma 3.1 that we may assume, without loss of generality, the existence of a 1-form θ in $\pi_D^{-1}(V_y)$ invariant under the action of $\mathbb{R}^d$, such that $\omega \mid \pi_D^{-1}(V_y) = d\theta$ and $\theta(v^j) > 0$ for each $j \in \{k+1, \ldots, l\}$. By Lemma 3.2, for each $j \in \{k+1, \ldots, l\}$, the vector field ξ^j is the Hamiltonian vector field of $\theta(\xi^j)$. Since ξ^j is the Hamiltonian vector field of g^j, the function $\theta(\xi^j) - g^j$ is constant along $D \mid \pi_D^{-1}(V)$ and V is contractible, it follows that there exists a constant b^j such that $g^j = \theta(\xi^j) + b^j$. Moreover,

$$\lim_{m\to\infty} [\xi^j(x_m)/|\varphi^j(x_m)|] = v^j, \theta(v^j) > 0 \text{ and } \lim_{m\to\infty} |\varphi^j(x_m)| = \infty$$

imply that $|\theta(\xi^j(x_m))| \to \infty$ as $x_m \to x$. Hence, the function $|g^j(x')|$ is unbounded as $x' \to x$. This completes the proof of the lemma. ∎

Theorem 2.5 states that, for a complete strongly admissible polarization F such that $F = \bar{F}$, dim $\mathscr{H} > 0$ if and only if $S \neq \emptyset$. Clearly, $S = \emptyset$ implies $\mathscr{H} = 0$. Suppose that $S \neq \emptyset$ and let k be the largest integer such that $S_k \neq \emptyset$. We shall show that $\mathscr{H}_0{}^k \neq \emptyset$. If $S_k \cap \operatorname{Int} X_k \neq \emptyset$, then there exists an open set V in X/D and an integral manifold Q of $T\pi_D({}_V K^\perp)$ such that $S_k \cap \operatorname{Int} X_k \cap \pi_D^{-1}(Q) \neq \emptyset$ and $X_k \cap \pi_D^{-1}(Q) \subseteq S_k$. Therefore, $S_k \cap \pi_D^{-1}(Q)$ has nonempty interior in $\pi_D^{-1}(Q)$. Let $\lambda \otimes \nu$ be a smooth section of $(L \otimes \sqrt{\wedge^n F}) \mid \pi_D^{-1}(Q)$ covariant constant along $F \mid Q$ with nonempty support contained in $S_k \cap \pi_D^{-1}(Q)$ which projects to a compact set in X/E. Such a section exists because the condition $F = \bar{F}$ implies that F is the complexification of D, and therefore $\lambda \otimes \nu$ is covariant constant along F if and only if it is covariant constant along D. Moreover, there exists a nonzero covariant constant along D sections of $(L \otimes \sqrt{\wedge^n F}) \mid \pi_D^{-1}(Q)$ provided $S_k \cap \pi_D^{-1}(Q)$ has a nonempty interior in $\pi_D^{-1}(Q)$. The condition that the projection to X/E of the support of $\lambda \otimes \nu$ is compact and the assumption $F = \bar{F}$, which is equivalent to $E = D$, imply that the integral (2.6) converges. Since the support of $\lambda \otimes \nu$ is not empty, $\lambda \otimes \nu \neq 0$, so that $\mathscr{H}_0{}^k \neq 0$. Hence dim $\mathscr{H} \geqslant$ dim $\mathscr{H}_0{}^k > 0$, which completes the proof of Theorem 2.5.

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